

ADELIC MAASS SPACES ON $U(2, 2)$

KRZYSZTOF KLOSIN

ABSTRACT. Generalizing the results of [6], [3] and [7], we define an adelic version of the Maass space for hermitian modular forms of weight k regarded as functions on adelic points of the quasi-split unitary group $U(2, 2)$ associated with an imaginary quadratic extension F/\mathbf{Q} of discriminant D_F . When the class number h_F of F is odd, we show that the Maass space is invariant under the action of the local Hecke algebras of $U(2, 2)(\mathbf{Q}_p)$ for all $p \nmid D_F$. As a consequence we obtain a Hecke-equivariant injective map from the Maass space to the h_F -fold direct product of the space of elliptic modular forms $M_{k-1}(\Gamma_0(D_F))$.

1. INTRODUCTION

In 1977 Saito and Kurokawa [8] conjectured that cusp forms in $S_{2k-2}(\mathrm{SL}_2(\mathbf{Z}))$ can be lifted to Siegel modular forms of weight k and that this lifting is Hecke-equivariant in an appropriate sense. This conjecture was proved in a series of papers by Maass [9], Andrianov [1], and Zagier [13], and later reformulated and proved using the language of representation theory by Piatetski-Shapiro [10]. In the 1980s Kojima [6] and Gritsenko [3] proved the existence of a similar lifting from the space of modular forms of level 4 and non-trivial character to the space of hermitian modular forms. The group $U(2, 2)$ for which the hermitian modular forms were defined was associated with the extension $\mathbf{Q}(i)/\mathbf{Q}$. Following [6] we will refer to the image of this lifting as the *Maass space* and to the lifting itself as the *Maass lifting*. In 1991 Krieg [7] extended the results of [6] and [3] to imaginary quadratic fields and showed Hecke-equivariance of the Maass lifting for a certain family of Hecke operators living in the local Hecke algebra at p , for p any inert prime. As [6], [3] and [7] all treated hermitian modular forms as classical objects (i.e., as functions defined on a higher-dimensional analogue of the complex upper half-plane), their methods did not yield Hecke-equivariance of the lift under the action of local Hecke algebras at split primes, except when the class number of F was one ([3]).

In this article we define an adelic version of the Maass space by imposing a condition on the Fourier coefficients of hermitian modular forms regarded as functions on $U(2, 2)(\mathbf{A})$ (Definition 5.1). In fact our definition reduces to that of Krieg's when the class number of F is one. We also show that the adelic Maass space, which we denote by \mathcal{M}_k , is invariant under the action of local Hecke algebras at all primes p not ramified in F/\mathbf{Q} (Theorem 6.1).

To prove Hecke-equivariance of the Maass space we assume that the class number h_F of F is odd, and in that case we show that \mathcal{M}_k is isomorphic to h_F copies of Krieg's Maass space (Proposition 5.9), hence we obtain a lifting from

$M_{k-1}(D_F, \chi_F)^{h_F}$ to \mathcal{M}_k together with a homomorphism from the Hecke algebra of \mathcal{M}_k into the space of endomorphisms of $M_{k-1}(D_F, \chi_F)^{h_F}$ generated by the classical Hecke operators T_p (for p split), T_p^2 (for p inert) and the group of permutations of the factors in $M_{k-1}(D_F, \chi_F)^{h_F}$ (Theorems 7.3 and 7.4).

In [5] the author used Hecke-equivariance of the Maass lifting for the extension $\mathbf{Q}(i)/\mathbf{Q}$ proved in [3] to construct congruences between Maass forms and non-Maass forms and as a consequence give evidence for the Bloch-Kato conjecture (Theorem 7.11 and Corollary 9.9 in [5]). We hope to be able to use Theorem 6.1 to extend the results of [5] to all imaginary quadratic fields of odd class number.

We believe that our assumption on the class number of F is unnecessary and we make it here only to make the proofs simpler. We work with the Hecke algebra for the algebraic group $\mathrm{U}(2, 2)$ instead of $\mathrm{GU}(2, 2)$ which is the case in [3] and [7], but our results easily imply analogous ones for $\mathrm{GU}(2, 2)$. Finally we want to point out that the Maass lifting we obtain agrees with a lifting that has recently been defined by Ikeda [4] using a different approach. Our result provides an explicit description of the image of that lifting as well as explicit formulas for the descent of Hecke algebras.

2. DEFINITIONS

Let F be an imaginary quadratic field with ring of integers \mathcal{O}_F . We denote by Cl_F the class group of F and set $h_F := \# \mathrm{Cl}_F$. For any affine group scheme X over \mathcal{O}_F and any \mathbf{Z} -algebra A , we denote by $x \mapsto \bar{x}$ the automorphism of $(\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} X)(A)$ induced by the non-trivial automorphism of F/\mathbf{Q} . Note that $(\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} X)(A)$ can be identified with a subgroup of $\mathrm{GL}_n(A \otimes \mathcal{O}_F)$ for some n . In what follows we always specify such an identification. Then for $x \in (\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} X)(A)$ we write x^t for the transpose of x , and set $x^* := \bar{x}^t$ and $\hat{x} := (\bar{x}^t)^{-1}$. Moreover, we write $\mathrm{diag}(a_1, a_2, \dots, a_n)$ for the $n \times n$ -matrix with a_1, a_2, \dots, a_n on the diagonal and all the off-diagonal entries equal to zero.

To the extension F/\mathbf{Q} we associate the quasi-split unitary group

$$\mathrm{U}(n, n) := \{g \in \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} \mathrm{GL}_{2n/\mathcal{O}_F} \mid gJ_n g^* = J_n\},$$

where $J_n = \begin{bmatrix} & -I_n \\ I_n & \end{bmatrix}$ and I_n stands for the $n \times n$ identity matrix. For $q \in \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} \mathrm{GL}_{n/\mathcal{O}_F}$, we set $p_q := \begin{bmatrix} q & \\ & \hat{q} \end{bmatrix} \in \mathrm{U}(n, n)$. We will also write G for $\mathrm{U}(2, 2)$ and J for J_2 .

For a prime ideal \mathfrak{p} of F , let $F_{\mathfrak{p}}$ denote the completion of F with respect to the valuation induced by \mathfrak{p} . Set $F_p := F \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and $\mathcal{O}_{F,p} = \mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Note that if p is inert or ramified in F , then F_p/\mathbf{Q}_p is a degree two extension of local fields and $a \mapsto \bar{a}$ induces the non-trivial automorphism in $\mathrm{Gal}(F_p/\mathbf{Q}_p)$, while if p splits in F , then $F_p \cong \mathbf{Q}_p \times \mathbf{Q}_p$, and $a \mapsto \bar{a}$ corresponds on the right-hand side to the automorphism defined by $(a, b) \mapsto (b, a)$. We denote the isomorphism $\mathbf{Q}_p \times \mathbf{Q}_p \xrightarrow{\sim} F_p$ by ι_p . For a matrix $g = (g_{ij})$ with entries in $\mathbf{Q}_p \times \mathbf{Q}_p$ we also set $\iota_p(g) = ((\iota_p(g_{ij})))$. For a split prime p the map ι_p^{-1} identifies $G(\mathbf{Q}_p)$ with

$$G_p = \{(g_1, g_2) \in \mathrm{GL}_4(\mathbf{Q}_p) \times \mathrm{GL}_4(\mathbf{Q}_p) \mid g_1 J g_2^t = J\}.$$

Note that the map $(g_1, g_2) \mapsto g_1$ gives a (non-canonical) isomorphism $G(\mathbf{Q}_p) \cong \mathrm{GL}_4(\mathbf{Q}_p)$.

Denote by \mathbf{A} (resp. \mathbf{A}_f , \mathbf{A}_F , $\mathbf{A}_{F,f}$) the ring of adeles of \mathbf{Q} (resp. finite adeles of \mathbf{Q} , adeles of F , finite adeles of F). For an adele a , we write a_f for its finite part.

Set $\hat{\mathbf{Z}} := \prod_p \mathbf{Z}_p$, $\hat{\mathcal{O}}_F := \mathcal{O}_F \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$, $K' = \mathrm{GL}_2(\hat{\mathcal{O}}_F)$ and $K = G(\hat{\mathbf{Z}})$. Note that K is a maximal compact subgroup of $G(\mathbf{A}_f)$. For any rational prime p , we denote by j_p the canonical embedding $G(\mathbf{Q}_p) \hookrightarrow G(\mathbf{A})$.

Let M_n denote the additive group of $n \times n$ matrices. Set

$$\begin{aligned} S &= \{h \in \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} M_{2/\mathcal{O}_F} \mid h^* = h\}, \\ T &= \{h \in S(\mathbf{Q}) \mid \mathrm{tr}(S(\mathbf{Z})h) \subset \mathbf{Z}\}, \\ T_p &= \{h \in S(\mathbf{Q}_p) \mid \mathrm{tr}(S(\mathbf{Z}_p)h) \subset \mathbf{Z}_p\}, \end{aligned}$$

and

$$T_{\mathbf{A}} = \prod_p T_p \subset M_2(\mathbf{A}_f).$$

For a matrix $h = (h_p)_p \in T_{\mathbf{A}}$, and a prime p , set

$$\epsilon_p(h) = \max\{n \in \mathbf{Z} \mid \frac{1}{p^n} h_p \in T_p\}$$

and

$$\epsilon(h) = \prod_p p^{\epsilon_p(h)}.$$

Note that since $h \in T_{\mathbf{A}}$, we have $\epsilon_p(h) \geq 0$ for every p .

3. SOME COSET DECOMPOSITIONS

To shorten notation in this section we put $U_n = U(n, n)$. It is well-known (see e.g., [2], Theorem 3.3.1) that for any finite subset \mathcal{B} of $\mathrm{GL}_n(\mathbf{A}_{F,f})$ of cardinality h_F with the property that the canonical homomorphism c_F defined as the composite $\mathbf{A}_F^\times \rightarrow \mathbf{A}_F^\times / F^\times \mathbf{C}^\times \hat{\mathcal{O}}_F^\times \cong \mathrm{Cl}_F$ restricted to $\det \mathcal{B}$ is a bijection, the following decomposition holds

$$\mathrm{GL}_n(\mathbf{A}_F) = \bigsqcup_{b \in \mathcal{B}} \mathrm{GL}_n(F) \mathrm{GL}_n(\mathbf{C}) b \mathrm{GL}_n(\hat{\mathcal{O}}_F).$$

We will call any such \mathcal{B} a *base*. We always assume that a base comes with a fixed ordering, so in particular if we consider a tuple $(f_b)_{b \in \mathcal{B}}$ indexed by elements of \mathcal{B} , and apply a permutation σ to the elements f_b , we do not consider the tuples $(f_b)_{b \in \mathcal{B}}$ and $(f_{\sigma^{-1}(b)})_{b \in \mathcal{B}}$ to be the same.

Put

$$H := \{a \in \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} \mathbf{G}_{m/\mathcal{O}_F} \mid a\bar{a} = 1\},$$

and let $\varphi : \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}} \mathbf{G}_{m/\mathcal{O}_F} \rightarrow H$ be defined by $\varphi(a) = a/\bar{a}$. Here \mathbf{G}_m denotes the multiplicative group.

Lemma 3.1. *For every rational prime p , we have $\det U_n(\mathbf{Z}_p) = \varphi(\mathcal{O}_{F,p}^\times)$.*

Proof. We have $\det U_n(\mathbf{Z}_p) \subset \varphi(\mathcal{O}_{F,p}^\times)$ by [12], Lemma 5.11(4). We will show the other containment. Every element of $\varphi(\mathcal{O}_{F,p}^\times)$ can be written as a/\bar{a} for some $a \in \mathcal{O}_{F,p}^\times$. For such an a , set $x_a := \mathrm{diag}(a, 1, 1, \dots, \bar{a}^{-1}, 1, 1, \dots)$, where \bar{a}^{-1} is on the $(n+1)$ st position. We certainly have $x_a \in U_n(\mathbf{Z}_p)$ and $\det x_a = a/\bar{a}$. \square

Proposition 3.2. *Assume $2 \nmid h_F$. For any base \mathcal{B} the following decomposition holds*

$$(3.1) \quad U_n(\mathbf{A}) = \bigsqcup_{b \in \mathcal{B}} U_n(\mathbf{Q}) U_n(\mathbf{R}) p_b U_n(\hat{\mathbf{Z}}).$$

Proof. First note that $\{c_F(\det p_b)\}_{b \in \mathcal{B}} = \text{Cl}_F$ as long as $2 \nmid h_F$. [Indeed, $\det p_q = \det q / \overline{\det q}$. Since $N_{F/\mathbf{Q}} \det q \in \mathbf{A}^\times \subset F^\times \mathbf{C}^\times \hat{\mathcal{O}}_F^\times$, i.e., $\det q / \overline{\det q}$ represents a principal fractional ideal of F , we have $c_F(\overline{\det q}) = c_F(\det q^{-1})$ and hence $c_F(\det p_b) = c_F(\det b)^2$. Since $2 \nmid h_F$, $\alpha \mapsto \alpha^2$ is an automorphism of Cl_F . Hence $c_F(\det \mathcal{B}) = \text{Cl}_F$ implies $\{c_F(\det p_b)\}_{b \in \mathcal{B}} = \text{Cl}_F$.] In the notation of Lemma 8.14 of [12], we have $D = U_n(\mathbf{R})U_n(\hat{\mathbf{Z}})$. By part (3) of that lemma, the map $x \mapsto \det x$ defines a bijection between $U_n(\mathbf{Q}) \setminus U_n(\mathbf{A})/D$ and $H(\mathbf{A})/H(\mathbf{Q})\det D$. Set U_0 (in the notation of part (4) of that lemma) to be $\mathbf{C}^\times \hat{\mathcal{O}}_F^\times$. Then part (4) of that lemma asserts that φ gives an isomorphism of $\mathbf{A}_F^\times / \mathbf{A}^\times F^\times U_0$ with $H(\mathbf{A})/H(\mathbf{Q})\varphi(U_0)$. By Lemma 3.1 we have $\varphi(U_0) = \det D$. Thus the composite of $x \mapsto \det x$ with φ gives a bijection of $U_n(\mathbf{Q}) \setminus U_n(\mathbf{A})/D$ onto $\mathbf{A}_F^\times / \mathbf{A}^\times F^\times U_0$, which in turn can be identified with Cl_F since $\mathbf{A}^\times = \mathbf{Q}^\times \times \mathbf{R}_+ \times \hat{\mathbf{Z}}^\times \subset F^\times U_0$. \square

Corollary 3.3. *If $(h_F, 2n) = 1$ a base \mathcal{B} can be chosen so that for all $b \in \mathcal{B}$ the matrices b and p_b are scalar matrices and $bb^* = b^*b = I_n$.*

Proof. It follows from the Tchebotarev density theorem, that elements of Cl_F can be represented by prime ideals. Since all the inert ideals are principal, Cl_F can be represented by prime ideals lying over split primes of the form $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Let Σ be a representing set consisting of such ideals \mathfrak{p} . As $(2n, h_F) = 1$, the set Σ^{2n} consisting of elements of Σ raised to the power $2n$ is also a representing set for Cl_F . Moreover, as $\mathfrak{p}\bar{\mathfrak{p}}$ is a principal ideal, $\bar{\mathfrak{p}} = \mathfrak{p}^{-1}$ as elements of Cl_F , hence $\Sigma' := \{\mathfrak{p}^n \bar{\mathfrak{p}}^{-n}\}_{\mathfrak{p} \in \Sigma}$ also represents all the elements Cl_F . Elements of Σ' can be written adelically as $\alpha_{\mathfrak{p}}^n$, with $\alpha_{\mathfrak{p}} = (1, 1, \dots, 1, p, p^{-1}, 1, \dots) \in \mathbf{A}_{F,f}$, where p appears on the \mathfrak{p} -th place and p^{-1} appears at the $\bar{\mathfrak{p}}$ -th place. Set $b_{\mathfrak{p}} = \alpha_{\mathfrak{p}} I_n$. Then we can take $\mathcal{B} = \{b_{\mathfrak{p}}\}_{\mathfrak{p}^n \bar{\mathfrak{p}}^{-n} \in \Sigma'}$ and we have $p_{b_{\mathfrak{p}}} = \alpha_{\mathfrak{p}} I_{2n}$. It is also clear that $bb^* = b^*b = I_n$. \square

4. HERMITIAN MODULAR FORMS

From now on let k be a positive integer divisible by $\#\mathcal{O}_F^\times$. Let $\mathbf{i} = [\begin{smallmatrix} i \\ i \end{smallmatrix}]$ and set

$$\mathcal{H} := \{Z \in M_2(\mathbf{C}) \mid -\mathbf{i}(Z - Z^*) > 0\}.$$

The group $G(\mathbf{R})$ acts on \mathcal{H} - an element $g = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}] \in G(\mathbf{R})$ (with $A, B, C, D \in M_2(\mathbf{C})$) sends $Z \in \mathcal{H}$ to $gZ := (AZ + B)(CZ + D)^{-1}$. Set $j(g, Z) := \det(CZ + D)$. For a congruence subgroup $\Gamma \subset G(\mathbf{Q})$, define $\mathcal{M}'_k(\Gamma)$ to be the \mathbf{C} -space consisting of functions $\phi : \mathcal{H} \rightarrow \mathbf{C}$ satisfying

$$(\phi|_k \gamma)(Z) := j(\gamma, Z)^{-k} \phi(\gamma Z) = \phi(Z)$$

for all $\gamma \in \Gamma$. Denote by $\mathcal{M}_k(\Gamma)$ the subspace of $\mathcal{M}'_k(\Gamma)$ consisting of holomorphic functions. We call elements of $\mathcal{M}_k(\Gamma)$ the Γ -hermitian modular forms (of weight k).

Let \mathcal{M}'_k denote the \mathbf{C} -space consisting of functions $f : G(\mathbf{A}) \rightarrow \mathbf{C}$ satisfying the following conditions:

- $f(\gamma g) = f(g)$ for all $\gamma \in G(\mathbf{Q})$, $g \in G(\mathbf{A})$,
- $f(g\kappa) = f(g)$ for all $\kappa \in K$, $g \in G(\mathbf{A})$,
- $f(hg) = j(h_\infty, g_\infty \mathbf{i}) f(g)$ for all $g = \gamma g_\infty \kappa \in G(\mathbf{Q})G(\mathbf{R})K$, $h = (h_\infty, 1) \in G(\mathbf{R})G(\mathbf{A}_f)$.

Let $\mathcal{C} \subset G(\mathbf{A}_f)$ be a finite subset such that

$$G(\mathbf{A}) = \bigsqcup_{c \in \mathcal{C}} G(\mathbf{Q})G(\mathbf{R})cK.$$

In particular if h_F is odd we can take $\mathcal{C} = \{p_b\}_{b \in \mathcal{B}}$ for any base \mathcal{B} . Let $f \in \mathcal{M}'_k$. For $g \in G(\mathbf{A})$, write $g = \gamma g_\infty c \kappa \in G(\mathbf{Q})G(\mathbf{R})cK$ for a unique $c \in \mathcal{C}$ and set $Z := g_\infty \mathbf{i}$. Set $f_c(Z) = j(g_\infty, \mathbf{i})^k f(cg)$. The map $f \mapsto (f_c)_{c \in \mathcal{C}}$ defines a \mathbf{C} -linear isomorphism $\Phi_{\mathcal{C}} : \mathcal{M}'_k \xrightarrow{\sim} \prod_{c \in \mathcal{C}} \mathcal{M}'_k(\Gamma_c)$, where $\Gamma_c := G(\mathbf{Q}) \cap (G(\mathbf{R})cKc^{-1})$ (cf. [12], p. 80). If h_F is odd, \mathcal{B} is a base and $\mathcal{C} = \{p_b\}_{b \in \mathcal{B}}$, we write Γ_b instead of Γ_{p_b} and f_b instead of f_{p_b} for $b \in \mathcal{B}$, and $\Phi_{\mathcal{B}}$ instead of $\Phi_{\mathcal{C}}$.

Definition 4.1. A function $f \in \mathcal{M}'_k$ whose image under the isomorphism Φ lands in $\prod_{c \in \mathcal{C}} \mathcal{M}_k(\Gamma_c)$ will be called a *hermitian modular form of weight k* . The space of hermitian modular forms of weight k will be denoted by \mathcal{M}_k .

We clearly have

$$(4.1) \quad \mathcal{M}_k \cong \prod_{c \in \mathcal{C}} \mathcal{M}_k(\Gamma_c).$$

For every $q \in G(\mathbf{A}_F)$, $f_q \in \mathcal{M}_k(\Gamma_q)$ possesses a Fourier expansion

$$f_q(Z) = \sum_{h \in S(\mathbf{Q})} c_q(h) e^{2\pi i \text{tr}(hZ)}.$$

Similarly, every $f \in \mathcal{M}_k$ possesses a Fourier expansion, i.e., for every $q \in \text{GL}_2(\mathbf{A}_F)$, and every $h \in S(\mathbf{Q})$ there exists a complex number $c_f(h, q)$ such that one has

$$f \left(\begin{bmatrix} I_2 & \sigma \\ & I_2 \end{bmatrix} \begin{bmatrix} q \\ \hat{q} \end{bmatrix} \right) = \sum_{h \in S(\mathbf{Q})} c_f(h, q) e_{\mathbf{A}}(\text{tr } h\sigma)$$

for every $\sigma \in S(\mathbf{A})$. Here $e_{\mathbf{A}}$ is defined in the following way. Let $a = (a_v) \in \mathbf{A}$, where v runs over all the places of \mathbf{Q} . If $v = \infty$, set $e_v(a_v) = e^{2\pi i a_v}$. If $v = p$, set $e_v(a_v) = e^{-2\pi i y}$, where y is a rational number such that $a_v - y \in \mathbf{Z}_p$. Then we set $e_{\mathbf{A}}(a) = \prod_v e_v(a_v)$.

Definition 4.2. Let \mathcal{B} be a base. We will say that $q \in \text{GL}_2(\mathbf{A}_F)$ belongs to a class $b \in \mathcal{B}$ if there exist $\gamma \in \text{GL}_2(F)$, $q_\infty \in \text{GL}_2(\mathbf{C})$ and $\kappa \in K'$ such that $q = \gamma b q_\infty \kappa$.

Remark 4.3. It is clear that the class of q depends only on $q_{\mathbf{f}}$.

Lemma 4.4. Suppose $r \in \text{GL}_2(\mathbf{A}_F)$ and $q \in \text{GL}_2(\mathbf{A}_F)$ belong to the same class and $r_{\mathbf{f}} = \gamma q_{\mathbf{f}} \kappa \in \text{GL}_2(F) q_{\mathbf{f}} \text{GL}_2(\hat{\mathcal{O}}_F)$. Then

$$(4.2) \quad c_f(h, r) = \left(\overline{\det r_\infty} / \overline{\det q_\infty} \right)^k e^{-2\pi i \text{tr}(r_\infty^* h r_\infty - q_\infty^* h q_\infty)} (\det \gamma^*)^{-k} c_f(\gamma^* h \gamma, q).$$

Proof. It follows from the proof of part (4) of Proposition 18.3 of [12], that

$$(4.3) \quad c_f(h, r) = \left(\overline{\det r_\infty} \right)^k e^{-2\pi i \text{tr}(r_\infty^* h r_\infty)} c_{p_{r_{\mathbf{f}}}}(h),$$

where

$$f_{p_{r_{\mathbf{f}}}}(Z) = \sum_{h \in S} c_{p_{r_{\mathbf{f}}}}(h) e^{2\pi i \text{tr } hZ}.$$

As is easy to see (cf. for example the Proof of Lemma 10.8 in [12]), $f_{p_{r_{\mathbf{f}}}} = f_{p_{q_{\mathbf{f}}}}|_k \begin{bmatrix} \gamma^{-1} & \\ & \gamma^* \end{bmatrix}$. Hence

$$(4.4) \quad c_{p_{r_{\mathbf{f}}}}(h) = (\det \gamma^*)^{-k} c_{p_{q_{\mathbf{f}}}}(\gamma^* h \gamma).$$

The Lemma follows from combining (4.3) with (4.4). \square

5. THE MAASS SPACE

Definition 5.1. Let \mathcal{B} be a base. We say that $f \in \mathcal{M}_k$ is a \mathcal{B} -Maass form if there exist functions $c_{b,f} : \mathbf{Z}_{\geq 0} \rightarrow \mathbf{C}$, $b \in \mathcal{B}$, such that for every $q \in \mathrm{GL}_2(\mathbf{A}_F)$ and every $h \in S(\mathbf{Q})$ the Fourier coefficient $c_f(h, q)$ satisfies

$$(5.1) \quad c_f(h, q) = (\overline{\det q_\infty})^k e^{-2\pi \mathrm{tr}(q_\infty^* h q_\infty)} (\det \gamma_{b,q}^*)^{-k} \times \\ \times \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(q_\mathbf{f}^* h q_\mathbf{f})}} d^{k-1} c_{b,f} \left(D_F d^{-2} \det h \prod_p p^{\mathrm{val}_p(\det q_\mathbf{f}^* q_\mathbf{f})} \right),$$

where $q_\mathbf{f} = \gamma_{b,q} b \kappa_q \in \mathrm{GL}_2(F) b K'$ for a unique $b \in \mathcal{B}$.

Remark 5.2. Note that by [12], Proposition 18.3(2), $c_f(h, q) \neq 0$ only if $(q^* h q)_p \in T_p$, so $\epsilon_p(q_\mathbf{f}^* h q_\mathbf{f}) \geq 0$. Also, note that Definition 5.1 is independent of the decomposition $q_\mathbf{f} = \gamma_{b,q} b \kappa_q \in \mathrm{GL}_2(F) b K'$. Indeed, if $q_\mathbf{f} = \gamma'_{b,q} b \kappa'_q \in \mathrm{GL}_2(F) b K'$ is another decomposition of $q_\mathbf{f}$, then

$$\det \gamma'_{b,q} \det \gamma_{b,q}^{-1} = \det(\kappa_q(\kappa'_q)^{-1}) \in \hat{\mathcal{O}}_F^\times \cap F^\times = \mathcal{O}_F^\times,$$

so $\det(\gamma'_{b,q})^k = \det \gamma_{b,q}^k$.

Definition 5.3. The \mathbf{C} -subspace of \mathcal{M}_k consisting of \mathcal{B} -Maass forms will be called the \mathcal{B} -Maass space.

Let $T'_\mathbf{A}$ (resp. T'_p) be defined in the same way as $T_\mathbf{A}$ (resp. T_p) except we do not require that $h \in T'_\mathbf{A}$ (resp. T'_p) be hermitian. Define ϵ' , ϵ'_p in the same way as ϵ , ϵ_p except replace T with T' . Then $\epsilon = \epsilon'|_T$.

Lemma 5.4. If $h = (h_p)_p \in T'_\mathbf{A}$ and $\kappa = (\kappa_p)_p \in K'$, then

$$\epsilon'(h\kappa) = \epsilon'(\kappa h) = \epsilon'(h).$$

Proof. If $h_p = p^n h'_p$, $h'_p \in T'_p$, then $\kappa_p h_p = p^n \kappa_p h'_p \in p^n T'_p$. On the other hand, if $\kappa_p h_p \in p^n T'_p$, then $h_p = \kappa_p^{-1}(\kappa_p h_p) \in p^n T'_p$ by the above argument. So, $h_p \in p^n T'_p$ if and only if $\kappa_p h_p \in p^n T'_p$, so $\epsilon_p(h\kappa) = \epsilon_p(h)$. The other equality is proved in the same way. \square

Corollary 5.5. If $q \in \mathrm{GL}_2(\mathbf{A}_F)$ and $r \in \mathrm{GL}_2(\mathbf{A}_F)$ are in the same class, and $r_\mathbf{f} = \gamma q_\mathbf{f} \kappa \in \mathrm{GL}_2(F) q_\mathbf{f} K'$, then $\epsilon(r_\mathbf{f}^* h r_\mathbf{f}) = \epsilon(q_\mathbf{f}^* \gamma^* h \gamma q_\mathbf{f})$.

Proposition 5.6. Choose a base \mathcal{B} and let $f \in \mathcal{M}_k$. If there exist functions $c_{b,f}^* : \mathbf{Z}_{\geq 0} \rightarrow \mathbf{C}$, $b \in \mathcal{B}$, such that for every $b \in \mathcal{B}$ and every $h \in S(\mathbf{Q})$, the Fourier coefficient $c_f(h, b)$ satisfies condition (5.1) with $c_{b,f}^*$ in place of $c_{b,f}$, then f is a \mathcal{B} -Maass form and one has $c_{b,f} = c_{b,f}^*$ for every $b \in \mathcal{B}$.

Proof. Fix \mathcal{B} and $f \in \mathcal{M}_k$. Suppose there exist $c_{b,f}^*$ such that (5.1) is satisfied for all pairs (h, b) . Let $q = \gamma b x \kappa = (\gamma x, \gamma b \kappa) \in \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{A}_{F,\mathbf{f}})$, where $\gamma \in \mathrm{GL}_2(F)$, $x \in \mathrm{GL}_2(\mathbf{C})$ and $\kappa \in K'$. Then by Lemma 4.4,

$$c_f(h, q) = (\overline{\det q_\infty})^k e^{-2\pi \mathrm{tr}(q_\infty^* h q_\infty - \gamma^* h \gamma)} (\det \gamma^*)^{-k} c_f(\gamma^* h \gamma, b).$$

Since condition (5.1) is satisfied for (h, b) , we know that

$$c_f(h, b) = e^{-2\pi \mathrm{tr} h} \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(b^* h b)}} d^{k-1} c_{b,f}^* \left(D_F d^{-2} \det h \prod_p p^{\mathrm{val}_p(\det b^* b)} \right).$$

Thus

$$c_f(\gamma^* h \gamma, b) = e^{-2\pi \text{tr}(\gamma^* h \gamma)} \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(b^* \gamma^* h \gamma b)}} d^{k-1} c_{b,f}^* \left(D_F d^{-2} \det(\gamma^* h \gamma) \prod_p p^{\text{val}_p(\det b^* b)} \right).$$

So,

$$(5.2) \quad c_f(h, q) = (\overline{\det q_\infty})^k e^{-2\pi \text{tr}(q_\infty^* h q_\infty)} (\det \gamma^*)^{-k} \times \\ \times \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(b^* \gamma^* h \gamma b)}} d^{k-1} c_{b,f}^* \left(D_F d^{-2} \det h \det(\gamma^* \gamma) \prod_p p^{\text{val}_p(\det b^* b)} \right).$$

The claim now follows since $\epsilon(b^* \gamma^* h \gamma b) = \epsilon(q_\mathbf{f}^* h q_\mathbf{f})$ by Corollary 5.5 and $\det(\gamma^* \gamma) \in \mathbf{Q}_+^*$, so $\det(\gamma^* \gamma) = \prod_p p^{\text{val}_p(\det \gamma^* \gamma)}$. \square

Proposition 5.7. *If \mathcal{B} and \mathcal{B}' are two bases, then the \mathcal{B} -Maass space and the \mathcal{B}' -Maass space coincide, i.e., the notion of a Maass form is independent of the choice of the base.*

Proof. Let \mathcal{B} and \mathcal{B}' be two bases. Write $q_\mathbf{f} = \gamma_{b,q} b \kappa_\mathcal{B} = \gamma_{b',q} b' \kappa_{\mathcal{B}'}$ with $b \in \mathcal{B}$, $b' \in \mathcal{B}'$, $\gamma_{b,q}, \gamma_{b',q} \in \text{GL}_2(F)$ and $\kappa_\mathcal{B}, \kappa_{\mathcal{B}'} \in K'$. Suppose f is a \mathcal{B} -Maass form, i.e., there exist functions $c_{b,f}$ for $b \in \mathcal{B}$, such that for every q and h ,

$$c_f(h, q) = t \det(\gamma_{b,q}^*)^{-k} \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(q_\mathbf{f}^* h q_\mathbf{f})}} d^{k-1} c_{b,f}(s),$$

where $t = (\overline{\det q_\infty})^k e^{-2\pi \text{tr}(q_\infty^* h q_\infty)}$ and $s = D_F d^{-2} \det h \prod_p p^{\text{val}_p(\det q_\mathbf{f}^* q_\mathbf{f})}$. Our goal is to show that there exist functions $c_{b',f}$ for $b' \in \mathcal{B}'$, such that for every q and h ,

$$(5.3) \quad c_f(h, q) = t \det(\gamma_{b',q}^*)^{-k} \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(q_\mathbf{f}^* h q_\mathbf{f})}} d^{k-1} c_{b',f}(s).$$

We have

$$(5.4) \quad \det \gamma_{b,q} = \det \gamma_{b',q} \det(b' b^{-1}) \det(\kappa_\mathcal{B}^{-1} \kappa_{\mathcal{B}'}).$$

Since $\det(b' b^{-1})$ corresponds to a principal fractional ideal, say $(\alpha_{b,b'})$, under the map $((\alpha_{b,b'})_\mathbf{p}) \mapsto \prod_\mathbf{p} \mathbf{p}^{\text{val}_\mathbf{p}((\alpha_{b,b'})_\mathbf{p})}$, using [2], Theorem 3.3.1, we can write $\det(b' b^{-1}) = \alpha_{b,b'} \kappa_{b,b'} \in \mathbf{A}_{F,\mathbf{f}}^\times$ with $\kappa_{b,b'} \in \hat{\mathcal{O}}_F^\times$. Then it follows from (5.4) that

$$\beta := \kappa_{b,b'} \det(\kappa_\mathcal{B}^{-1} \kappa_{\mathcal{B}'}) \in \hat{\mathcal{O}}_F^\times \cap F^\times = \mathcal{O}_F^\times.$$

Hence $\beta^k = 1$. Thus $(\det \gamma_{b,q}^*)^{-k} = (\det \gamma_{b',q}^*)^{-k} \overline{\alpha_{b,b'}}^{-k}$. Note that $\alpha_{b,b'}^{-k}$ is well defined and only depends on b and b' (i.e., it is independent of q and h). Set $c_{b',f}(n) = \overline{\alpha_{b,b'}}^{-k} c_{b,f}(n)$. Then it is clear that $c_{b',f}$ satisfies (5.3). \square

Definition 5.8. From now on we will refer to \mathcal{B} -Maass forms simply as *Maass forms*. Similarly we will talk about the *Maass space* instead of \mathcal{B} -Maass spaces. This is justified by Proposition 5.7. The Maass space will be denoted by \mathcal{M}_k^M .

We now recall the definition of Maass space introduced in [7]. We will refer to it as the $G(\mathbf{Z})$ -Maass space. Consider the space $\mathcal{M}_k(G(\mathbf{Z}))$ of $G(\mathbf{Z})$ -hermitian modular forms of weight k . We say that $\phi(Z) = \sum_{h \in T} c_\phi(h) e^{2\pi i \text{tr}(hZ)} \in \mathcal{M}_k(G(\mathbf{Z}))$ is a $G(\mathbf{Z})$ -Maass form if there exists a function $\alpha_\phi : \mathbf{Z}_{\geq 0} \rightarrow \mathbf{C}$ such that for every $h \in T$, one has

$$(5.5) \quad c_\phi(h) = \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(h)}} d^{k-1} \alpha_\phi(D_F d^{-2} \det h).$$

The subspace of $\mathcal{M}_k(G(\mathbf{Z}))$ consisting of $G(\mathbf{Z})$ -Maass forms will be denoted by $\mathcal{M}_k^{\text{M}}(G(\mathbf{Z}))$.

Proposition 5.9. *If $2 \nmid h_F$, then the Maass space \mathcal{M}_k^{M} is isomorphic (as a \mathbf{C} -linear space) to $\#\mathcal{B}$ copies of the $G(\mathbf{Z})$ -Maass space $\mathcal{M}_k^{\text{M}}(G(\mathbf{Z}))$.*

Proof. Since the Maass space is independent of the choice of a base \mathcal{B} by Proposition 5.7, we may choose \mathcal{B} as in Corollary 3.3. The map $\Phi_{\mathcal{B}} : \mathcal{M}_k \rightarrow \prod_{b \in \mathcal{B}} \mathcal{M}_k(G(\mathbf{Z}))$ is an isomorphism. Let $f \in \mathcal{M}_k^{\text{M}}$ and set $(f_b)_{b \in \mathcal{B}} = \Phi_{\mathcal{B}}(f)$. Set $\alpha_{f_b} := c_{b,f}$. Then using (4.3), and the fact that the matrices b commute with h and $b^*b = 1$, we see that condition (5.1) for $c_f(h, b)$ translates into condition (5.5) for $c_{f_b}(h)$. Hence $\Phi_{\mathcal{B}}(\mathcal{M}_k^{\text{M}}) \subset \prod_{b \in \mathcal{B}} \mathcal{M}_k^{\text{M}}(G(\mathbf{Z}))$. On the other hand if $(f_b)_{b \in \mathcal{B}} \in \prod_{b \in \mathcal{B}} \mathcal{M}_k^{\text{M}}(G(\mathbf{Z}))$, set $c_{b,f} := \alpha_{f_b}$. Then conditions (5.5) for $c_{f_b}(h)$ translate into conditions (5.1) for $c_f(b, h)$. By Proposition 5.6 this implies that f is a Maass form. \square

6. INVARIANCE UNDER HECKE ACTION

It was proved in [7] that the $G(\mathbf{Z})$ -Maass space is invariant under the action of a certain Hecke operator T_p associated with a prime p which is inert in F . On the other hand Gritsenko in [3] proved the invariance of the $G(\mathbf{Z})$ -Maass space under all the Hecke operators when the class number of F is equal to 1. In this section we show that if the class number of F is odd, then the Maass space \mathcal{M}_k^{M} is in fact invariant under all the local Hecke algebras (for primes $p \nmid D_F$).

6.1. The Hecke algebra. From now on assume that h_F is odd. Let p be a rational prime and write K_p for $G(\mathbf{Z}_p)$. Let \mathcal{H}_p be the \mathbf{C} -algebra generated by double cosets $K_p g K_p$, $g \in G(\mathbf{Q}_p)$ with the usual law of multiplication (cf. [12], section 11). If $K_p g K_p \in \mathcal{H}_p$, there exists a finite set $A_g \subset G(\mathbf{Q}_p)$ such that $K_p g K_p = \bigsqcup_{\alpha \in A_g} K_p \alpha$.

For $f \in \mathcal{M}_k$, $g \in G(\mathbf{Q}_p)$, $h \in G(\mathbf{A})$, set

$$([K_p g K_p]f)(h) = \sum_{\alpha \in A_g} f(h j_p(\alpha)^{-1}).$$

It is clear that $[K_p g K_p]f \in \mathcal{M}_k$.

Theorem 6.1. *Let $p \nmid D_F$ be a rational prime. The Maass space is invariant under the action of \mathcal{H}_p , i.e., if $f \in \mathcal{M}_k^{\text{M}}$, and $g \in G(\mathbf{Q}_p)$, then $[K_p g K_p]f \in \mathcal{M}_k^{\text{M}}$.*

Proof. We will only present the proof in the case when p splits in F/\mathbf{Q} . For such a prime p the invariance of \mathcal{M}_k^{M} under the action of \mathcal{H}_p follows from Lemma 6.2 and Propositions 6.7, 6.10 and 6.11 below. If p is inert one can proceed along the same lines, however, it is the case when p splits that is essentially new. Indeed, if p is inert, the elements of \mathcal{H}_p respect the decomposition (4.1), hence the statement of the theorem reduces to an assertion about the action of \mathcal{H}_p on $\mathcal{M}_k(G(\mathbf{Z}))$. Then

the method used in [3] can be adapted to prove the theorem. See also Theorem 7 in [7] which proves the invariance of the $G(\mathbf{Z})$ -Maass space for a certain family of Hecke operators in \mathcal{H}_p . \square

Let p be a prime which splits in F . Write $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Recall that $G(\mathbf{Q}_p) \cong \mathrm{GL}_4(\mathbf{Q}_p)$, and an element A of $G(\mathbf{Q}_p)$ can be written as $A = (A_1, A_2) \in \mathrm{GL}_4(\mathbf{Q}_p) \times \mathrm{GL}_4(\mathbf{Q}_p)$ with $A_2 = -J(A_1^t)^{-1}J$. Note that if $a, b, c, d \in M_2(\mathbf{C})$ then

$$-J \begin{bmatrix} a & b \\ c & d \end{bmatrix} J = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Set

- $T_{\mathfrak{p}} := K_{p^t p}((\mathrm{diag}(p^{-1}, 1, 1, 1), \mathrm{diag}(1, 1, p, 1)))K_p$
- $U_{\mathfrak{p}} := K_{p^t p}((\mathrm{diag}(p^{-1}, p^{-1}, 1, 1), \mathrm{diag}(1, 1, p, p)))K_p$,
- $\Delta_{\mathfrak{p}} := K_{p^t p}((pI_4, p^{-1}I_4))K_p$.

Lemma 6.2. *The \mathbf{C} -algebra \mathcal{H}_p is generated by the operators $T_{\mathfrak{p}}$, $T_{\bar{\mathfrak{p}}}$, $U_{\mathfrak{p}}$, $\Delta_{\mathfrak{p}}$ and their inverses.*

Proof. This follows from the theory of Hecke algebras on $GL_4(\mathbf{Q}_p)$. \square

6.2. Diagonalizing hermitian matrices mod p^n . We begin by proving an analogue of Proposition 7 of [7] for a split prime p .

Proposition 6.3. *Let n be a positive integer, p a prime number split in F and write $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Let $h \in T$, $h \neq 0$. Then there exist $a, d \in \mathbf{Z}_+$ with $p \nmid a$ and $u \in \mathrm{SL}_2(\mathcal{O}_F)$ such that*

$$u^* h u \equiv \epsilon(h) \begin{bmatrix} a & \\ & d \end{bmatrix} \pmod{p^n T}.$$

In fact it is enough to prove the following lemma.

Lemma 6.4. *Proposition 6.3 holds if T is replaced by the subgroup of hermitian matrices inside $M_2(\mathcal{O}_F)$.*

Proof. Without loss of generality we may assume that $\epsilon(h) = 1$. Let (M, M^t) be the image of h under the composite

$$M_2(\mathcal{O}_F) \rightarrow M_2(\mathcal{O}_F/p^n) \xrightarrow{\sim} M_2(\mathcal{O}_F/\mathfrak{p}^n) \oplus M_2(\mathcal{O}_F/\bar{\mathfrak{p}}^n) \cong M_2(\mathbf{Z}/p^n) \oplus M_2(\mathbf{Z}/p^n).$$

Since the canonical map $\mathrm{SL}_2(\mathcal{O}_F) \rightarrow \mathrm{SL}_2(\mathcal{O}_F/p^n) \cong \mathrm{SL}_2(\mathbf{Z}/p^n) \oplus \mathrm{SL}_2(\mathbf{Z}/p^n)$ is surjective ([11], p. 490), it is enough to find $A_1, A_2 \in \mathrm{SL}_2(\mathbf{Z}/p^n \mathbf{Z})$ such that

$$(6.1) \quad A_2^t M A_1 = \begin{bmatrix} \alpha & \\ & \delta \end{bmatrix}$$

with $\alpha \neq 0 \pmod{p}$. The existence of such A_1 and A_2 is clear. \square

6.3. Invariance under $T_{\mathfrak{p}}$.

Lemma 6.5. *We have the following decomposition*

$$\begin{aligned}
 (6.2) \quad T_{\mathfrak{p}} = & \bigsqcup_{a,b,c \in \mathbf{Z}/p\mathbf{Z}} K_p \left(\begin{bmatrix} p^{-1} & ap^{-1} & bp^{-1} & cp^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & b \\ & 1 \\ & c \\ -a & 1 \end{bmatrix} \right) \sqcup \\
 & \bigsqcup_{d,e \in \mathbf{Z}/p\mathbf{Z}} K_p \left(\begin{bmatrix} 1 & & & \\ p^{-1} & dp^{-1} & ep^{-1} & \\ & 1 & & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & d \\ & 1 \\ & e \\ & p \end{bmatrix} \right) \sqcup \\
 & \bigsqcup_{f \in \mathbf{Z}/p\mathbf{Z}} K_p \left(\begin{bmatrix} 1 & & & \\ & 1 & p^{-1} & \\ & p^{-1} & p^{-1}f & \\ & & 1 & \end{bmatrix}, \begin{bmatrix} p & & & \\ -f & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \sqcup \\
 & K_p \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & p^{-1} \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right).
 \end{aligned}$$

Proof. This follows from an analogous decomposition for $\mathrm{GL}_4(\mathbf{Q}_p)$. \square

Let $g = T_{\mathfrak{p}}f$. Then for $q \in \mathrm{GL}_2(\mathbf{A}_F)$ and $\sigma \in S(\mathbf{A})$, we can write

$$g \left(\begin{bmatrix} q & \sigma \hat{q} \\ & \hat{q} \end{bmatrix} \right) = \sum_{h \in S} c_g(h, q) e_{\mathbf{A}}(\mathrm{tr}(h\sigma)).$$

Define the following matrices

$$\alpha'_a = \left(\begin{bmatrix} p & -a \\ & 1 \end{bmatrix}, I_2 \right) \in \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p) \quad a = 0, 1, \dots, p-1$$

and

$$\alpha'_p = \left(\begin{bmatrix} 1 & \\ & p \end{bmatrix}, I_2 \right) \in \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p).$$

For $a = 0, 1, \dots, p$, set $\alpha_a = \iota_p(\alpha'_a) \in \mathrm{GL}_2(F_p)$.

Lemma 6.6. *One has the following formula*

$$c_g(h, q) = p^2 \sum_{a=0}^p c_f(h, q\alpha_a) + \sum_{a=0}^p c_f(h, q\hat{\alpha}_a).$$

Proof. This is a straightforward calculation using Lemma 6.5. \square

Proposition 6.7. *The Maass space is invariant under the action of $T_{\mathfrak{p}}$, i.e., if $f \in \mathcal{M}_k^{\mathbf{M}}$, then $g \in \mathcal{M}_k^{\mathbf{M}}$.*

Proof. Choose a base \mathcal{B} as in Corollary 3.3. In particular we have $\epsilon(b^*hb) = \epsilon(h)$ and $\mathrm{val}_p(\det b^*b) = 0$. By Propositions 5.6 and 5.7 it is enough to show that there exist functions $c_{b,g} : \mathbf{Z}_+ \rightarrow \mathbf{C}$ ($b \in \mathcal{B}$) such that

$$(6.3) \quad c_g(h, b) = e^{-2\pi \mathrm{tr} h} \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(h)}} d^{k-1} c_{b,g}(D_F d^{-2} \det h).$$

For $b \in \mathcal{B}$, set $b' = b\alpha_p$. Note that all of the matrices: $b\alpha_a$, $b\hat{\alpha}_a$, ($a = 0, 1, \dots, p$) belong to the same class b' . Denote any of those matrices by q . Then $q = \gamma_{b',q} b' \kappa_q \in \mathrm{GL}_2(F)b'K'$ and it is easy to see that

$$\det \gamma_{b',q}^k = \begin{cases} 1 & q = b\alpha_a, \quad a = 0, 1, \dots, p \\ p^{-k} & q = b\hat{\alpha}_a, \quad a = 0, 1, \dots, p, \end{cases}$$

and

$$\text{val}_p(\det q^* q) = \begin{cases} 1 & q = b\alpha_a \quad a = 0, 1, \dots, p \\ -1 & q = b\hat{\alpha}_a \quad a = 0, 1, \dots, p. \end{cases}$$

Write $h = \epsilon(h)h'$. One has $\epsilon(h') = 1$. Set $D = D_F \det h$ and $D' = D_F \det h'$. Using Lemma 6.6 and the fact that f is a Maass form, we obtain

$$(6.4) \quad c_g(h, b) = e^{-2\pi \text{tr } h} \times \left(p^2 \sum_{a=0}^p \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(\alpha_a^* h \alpha_a)}} d^{k-1} c_{b',f}(Dd^{-2}p) + \right. \\ \left. + p^k \sum_{a=0}^p \sum_{\substack{d \in \mathbf{Z}_+ \\ d | \epsilon(\hat{\alpha}_a^* h \hat{\alpha}_a)}} d^{k-1} c_{b',f}(Dd^{-2}p^{-1}) \right).$$

Using Proposition 6.3, one can relate $\epsilon(\alpha_a^* h \alpha_a)$ and $\epsilon(\hat{\alpha}_a^* h \hat{\alpha}_a)$ to $\epsilon(h)$ for $a = 0, 1, \dots, p$, and then (6.4) becomes

$$(6.5) \quad c_g(h, b) = e^{-2\pi \text{tr } h} p^2 \sum_0 A_d^{(1)} + e^{-2\pi \text{tr } h} \times \\ \times \begin{cases} p^3 \sum_0 A_d^{(1)} & p \nmid D', p \nmid \epsilon(h) \\ p^k(p+1) \sum_{-1} A_d^{(-1)} + p^3 \sum_0 A_d^{(1)} & p \nmid D', p \mid \epsilon(h) \\ p^2(p-1) \sum_0 A_d^{(1)} + p^2 \sum_1 A_d^{(1)} + p^k \sum_0 A_d^{(-1)} & p \mid D', p \nmid \epsilon(h) \\ p^2(p-1) \sum_0 A_d^{(1)} + p^2 \sum_1 A_d^{(1)} + p^k \sum_0 A_d^{(-1)} + p^{k+1} \sum_{-1} A_d^{(-1)} & p \mid D', p \mid \epsilon(h), \end{cases}$$

where $\sum_n A_d^{(m)} = \sum_{d | p^n \epsilon(h)} A_d^{(m)}$, $A_d^{(m)} = d^{k-1} c_{b',f}(Dd^{-2}p^m)$.

For D in the image of the map $h \mapsto D_F \epsilon(h)^{-2} \det h$ and $b \in \mathcal{B}$ we make the following definition

$$(6.6) \quad c_{b,g}(D) = p^2(p+1) c_{b',f}(Dp) + p^k(p+1) c_{b',f}(Dp^{-1}),$$

where we assume that $c_{b',f}(n) = 0$ when $n \notin \mathbf{Z}_+$. If D is not in the image of that map, we set $c_{b,g}(D) = 0$. Note that we clearly have

$$c_g(h, b) = e^{-2\pi \text{tr } h} c_{b,g}(D_F \det h)$$

for every h with $\epsilon(h) = 1$. Thus to check if g lies in the Maass space we just need to check that (6.3) holds with $c_{b,g}$ defined by (6.6). This is an easy calculation using (6.5). \square

6.4. Invariance under U_p . This is completely analogous to the proof for T_p , hence we only include the relevant formulas for the reader's convenience.

Lemma 6.8. *We have the following decomposition:*

$$\begin{aligned}
 (6.7) \quad U_{\mathfrak{p}} = & \bigsqcup_{b,c,d,e \in \mathbf{Z}/p\mathbf{Z}} K_p \left(\begin{bmatrix} p^{-1} & bp^{-1} & dp^{-1} \\ & p^{-1} & cp^{-1} \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & b & c \\ & 1 & d \\ & & p \end{bmatrix} \right) \sqcup \\
 & \bigsqcup_{a,c,f \in \mathbf{Z}/p\mathbf{Z}} K_p \left(\begin{bmatrix} 1 & & \\ -fp^{-1} & p^{-1} & cp^{-1} \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & c \\ & p & 1 \\ & & f \end{bmatrix} \right) \sqcup \\
 & \bigsqcup_{e,f \in \mathbf{Z}/p\mathbf{Z}} K_p \left(\begin{bmatrix} 1 & & \\ -fp^{-1} & p^{-1} & ep^{-1} \\ & & 1 \end{bmatrix}, \begin{bmatrix} p & 1 & e \\ & 1 & f \\ & & p \end{bmatrix} \right) \sqcup \\
 & \bigsqcup_{a,b \in \mathbf{Z}/p\mathbf{Z}} K_p \left(\begin{bmatrix} p^{-1} & bp^{-1} \\ & 1 \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & b \\ & p & p \\ & & 1 \end{bmatrix} \right) \sqcup \\
 & \bigsqcup_{d \in \mathbf{Z}/p\mathbf{Z}} K_p \left(\begin{bmatrix} p^{-1} & & dp^{-1} \\ & 1 & \\ & & p^{-1} \\ & & & 1 \end{bmatrix}, \begin{bmatrix} p & 1 & d \\ & 1 & p \\ & & 1 \end{bmatrix} \right) \sqcup \\
 & K_p \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & & p^{-1} \end{bmatrix}, \begin{bmatrix} p & & \\ & p & \\ & & 1 \end{bmatrix} \right).
 \end{aligned}$$

As before, let $g = T_{\mathfrak{p}}f$. Define matrices:

$$\begin{aligned}
 \beta'_p &= \left(\begin{bmatrix} p \\ & p \end{bmatrix}, I_2 \right) \in \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p), \\
 \gamma'_a &= \left(\begin{bmatrix} 1 \\ a & p \end{bmatrix}, I_2 \right) \in \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p), \quad a = 0, 1, \dots, p-1, \\
 \gamma'_p &= \left(\begin{bmatrix} p \\ & 1 \end{bmatrix}, I_2 \right) \in \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p),
 \end{aligned}$$

and set $\beta_p = \iota_p(\beta'_p) \in \mathrm{GL}_2(F_p)$, $\gamma_a = \iota_p(\gamma'_a) \in \mathrm{GL}_2(F_p)$ ($a = 0, 1, \dots, p$).

Lemma 6.9. *One has the following formula*

$$c_g(h, q) = p^4 c_f(h, q\beta_p) + c_f(h, q\hat{\beta}_p) + p \sum_{a=0}^p \sum_{b=0}^p c_f(h, q\gamma_a \hat{\gamma}_b).$$

Proposition 6.10. *The Maass space is invariant under the action of $U_{\mathfrak{p}}$.*

Proof. This is similar to the proof of Proposition 6.7. Let b, D, D', h, h' be as in that proof. Then for all a, c , we see that $b\beta_p, b\hat{\beta}_p, b\gamma_a \hat{\gamma}_c$ all lie in the same class $b' = b\beta_p$. One has

$$\det \gamma_{b',q}^k = \begin{cases} 1 & q = b\beta_p \\ p^{-k} & q = b\gamma_a \hat{\gamma}_c, \quad a, c \in \{0, 1, \dots, p\} \\ p^{-2k} & q = b\hat{\beta}_p, \end{cases}$$

and

$$\mathrm{val}_p(\det q^* q) = \begin{cases} 2 & q = b\beta_p \\ 0 & q = b\gamma_a \hat{\gamma}_c, \quad a, c \in \{0, 1, \dots, p\} \\ -2 & q = b\hat{\beta}_p. \end{cases}$$

Using Proposition 6.3 as in the proof of Proposition 6.7 we obtain

$$(6.8) \quad c_g(h, b) = e^{-2\pi \text{tr } h} p^4 \sum_1 A_d^{(2)} + p^{2k} \sum_{-1} A_d^{(-2)} + p^{k+1}(p+1) \sum_0 A_d^{(0)} + p^{k+3} \sum_{-1} A_d^{(0)} + \\ + e^{-2\pi \text{tr } h} p^{k+1} \begin{cases} p \sum_{-1} A_d^{(0)} & p \nmid D' \\ p \sum_0 A_d^{(0)} & p \mid D', p^2 \nmid D' \\ \sum_1 A^{(0)} + (p-1) \sum_0 A^{(0)} & p^2 \mid D', \end{cases}$$

where if there is no d dividing $p^n \epsilon(h)$, we set $\sum_n = 0$. For D in the image of the map $h \mapsto D_F \epsilon(h)^{-2} \det h$, we make the following definition:

$$(6.9) \quad c_{b,g}(D) = p^4 c_{b',f}(Dp^2) + (p^{k+3} + p^{k+2} + p^{k+1}) c_{b',f}(D) + \\ + \begin{cases} 0 & p \nmid D \\ p^{k+2} c_{b',f}(D) & p \mid D, p^2 \nmid D \\ p^{k+2} c_{b',f}(D) + p^{2k} c_{b',f}(Dp^{-2}) & p^2 \mid D. \end{cases}$$

We now check as in the proof of Proposition 6.7 that g is a Maass form. \square

6.5. Invariance under Δ_p . Let $g = \Delta_p f$ and set $\delta_p = \iota_p((p^{-1}I_4, pI_4))$. Then we have

$$c_g(h, q) = c_f(h, q\delta_p).$$

Proposition 6.11. *The Maass space is invariant under the action of Δ_p .*

Proof. Let \mathcal{B} be as before. For $b \in \mathcal{B}$ set $b' = b\delta_p$. One clearly has $\text{val}_p(\delta_p^* \delta_p) = 0$ and $\epsilon(\delta_p^* h \delta_p) = \epsilon(h)$. Hence one can define

$$(6.10) \quad c_{b,g}(D) = c_{b',f}(D).$$

The claim is now clear. \square

7. DESCENT

Assume that h_F is odd. Let χ_F be the quadratic Dirichlet character attached to the extension F/\mathbf{Q} . For a positive integer n , set

$$a_F(n) = \#\{\alpha \in (iD_F^{-1/2}\mathcal{O}_F)/\mathcal{O}_F \mid D_F N_{F/\mathbf{Q}}(\alpha) \equiv -n \pmod{D_F}\}.$$

Let \mathcal{B} be a base as in Corollary 3.3.

Theorem 7.1. *There exists a \mathbf{C} -linear injection of vector spaces*

$$\text{Desc}_{\mathcal{B}} : \mathcal{M}_k^{\mathbf{M}} \hookrightarrow \prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F),$$

such that $\text{Desc}(f) = (F_b)_{b \in \mathcal{B}}$ with

$$a_{F_b}(n) = i \frac{a_F(n)}{\sqrt{D_F}} c_{b,f}(n),$$

where $a_{F_b}(n)$ is the n -th Fourier coefficient of F_b . The map $\text{Desc}_{\mathcal{B}}$ depends on the choice of \mathcal{B} .

Proof. This follows immediately from [7], Theorem 6 and formula (4) using our assumption on \mathcal{B} and (4.3). \square

Remark 7.2. Krieg in [7] explicitly describes the image of the descent map he defines and denotes it by $G_{k-1}(D_F, \chi_F)^*$. The image of Desc is exactly $\prod_{b \in \mathcal{B}} G_{k-1}(D_F, \chi_F)^* \subset \prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F)$.

Let $S_{\mathcal{B}}$ denote the group of permutations of \mathcal{B} . For $b \in \mathcal{B}$, let $\sigma_{\mathbf{p},n} \in S_{\mathcal{B}}$ be the unique permutation such that $b\alpha_p^n$ is in the class of $\sigma_{\mathbf{p},n}(b)$. If $A = (a_b)_{b \in \mathcal{B}}$ is an ordered tuple indexed by elements of \mathcal{B} , we define $\sigma_{\mathbf{p},n}A = (a_{\sigma_{\mathbf{p},n}^{-1}(b)})_{b \in \mathcal{B}}$. Moreover, for $b \in \mathcal{B}$ and $n \in \mathbf{Z}$, write $\gamma_{b,\mathbf{p},n}$ for an element of $G(\mathbf{Q})$ such that $b\alpha_p^n = \gamma_{b,\mathbf{p},n}\sigma_{\mathbf{p},n}(b)\kappa$ for $\kappa \in K$, and denote by $\gamma_{\mathcal{B},\mathbf{p},n}$ the \mathcal{B} -tuple $(\det \gamma_{b,\mathbf{p},n}^*)_{b \in \mathcal{B}}$.

For a rational prime $p \nmid D_F$ denote by T_p the operator acting on $\prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F)$ which sends $(F_b)_{b \in \mathcal{B}}$ to $(F'_b)_{b \in \mathcal{B}}$, where $F_b(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$, $F'_b(z) = \sum_{n=1}^{\infty} a'(n)e^{2\pi inz}$ with $a'(n) = a(np) + \chi_F(p)p^{k-2}a(n/p)$. Here $a(m) = 0$ if $m \notin \mathbf{Z}_{\geq 0}$. Denote by \mathbf{T}_p the \mathbf{C} -subalgebra of endomorphisms of $\prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F)$ generated by T_p , the group $S_{\mathcal{B}}$ and \mathcal{B} -tuples of complex numbers acting on $\prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F)$ in an obvious way.

Theorem 7.3. *Let p be a rational prime which splits in F/\mathbf{Q} . There exists a \mathbf{C} -algebra map*

$$\text{Desc}_{\mathcal{B},p} : \mathcal{H}_p \rightarrow \mathbf{T}_p,$$

such that for every $H \in \mathcal{H}_p$ the following diagram

$$\begin{array}{ccc} \mathcal{M}_k & \xrightarrow{H} & \mathcal{M}_k \\ \downarrow \text{Desc}_{\mathcal{B}} & & \downarrow \text{Desc}_{\mathcal{B}} \\ \prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F) & \xrightarrow{\text{Desc}_{\mathcal{B},p}(H)} & \prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F) \end{array}$$

commutes. Moreover, one has

$$\begin{aligned} \text{Desc}_{\mathcal{B},p}(T_{\mathbf{p}}) &= (\gamma_{\mathcal{B},\mathbf{p},1})^{-k} p^2 (p+1) T_p \circ \sigma_{\mathbf{p},1}, \\ \text{Desc}_{\mathcal{B},p}(U_{\mathbf{p}}) &= (\gamma_{\mathcal{B},\mathbf{p},2})^{-k} p^4 (T_p^2 + p^{k-1} + p^{k-3}) \circ \sigma_{\mathbf{p},2} \\ \text{Desc}_{\mathcal{B},p}(\Delta_{\mathbf{p}}) &= (\gamma_{\mathcal{B},\mathbf{p},4})^{-k} \sigma_{\mathbf{p},4}. \end{aligned} \tag{7.1}$$

Proof. This follows from Theorem 7.1, and formulas (6.6), (6.9), and (6.10). Just for illustration, we include the argument in the case of $T_{\mathbf{p}}$. Let $f \in \mathcal{M}_k^{\mathbf{M}}$ and set $g = T_{\mathbf{p}}f$. Fix $b \in \mathcal{B}$ and write b_1 for $\sigma_{\mathbf{p},1}(b)$ and b' for $b\alpha_p$. There exist diagonal matrices $\gamma \in G(\mathbf{Q})$ and $\kappa \in K$ such that $b' = \gamma b_1 \kappa$ (hence also $b_1 = \gamma^{-1} b' \kappa^{-1}$). Identify $\mathcal{M}_k^{\mathbf{M}}$ with $\prod_{b \in \mathcal{B}} \mathcal{M}_k^{\mathbf{M}}(G(\mathbf{Z}))$ via $f' \mapsto (f'_b)_{b \in \mathcal{B}}$. For $\phi \in \mathcal{M}_k^{\mathbf{M}}(G(\mathbf{Z}))$, $h \in T$, denote by $c_{\phi}(h)$ the h -Fourier coefficient of ϕ . We will study the action of $T_{\mathbf{p}}$ on $c_{f_{b_1}}(h)$. Since f is a Maass form it is enough to consider h of the form $\begin{bmatrix} 1 & * \\ & * \end{bmatrix}$. Fix such an h . Set $D = D_F \det h$. Then by (4.3) and (6.6),

$$c_{g_b}(h) = e^{2\pi \text{tr } h} c_g(h, b) = p^2 (p+1) (c_{b',f}(Dp) + p^{k-2} c_{b',f}(Dp^{-1})).$$

By (4.3) and (5.1), we have

$$\begin{aligned} c_{f_{b_1}}(h \begin{bmatrix} 1 & \\ & p \end{bmatrix}) &= e^{2\pi \text{tr } h \begin{bmatrix} 1 & \\ & p \end{bmatrix}} c_f(h \begin{bmatrix} 1 & \\ & p \end{bmatrix}, b_1) = c_{b_1,f}(Dp) = (\det \gamma^*)^k c_{b',f}(Dp) \\ c_{f_{b_1}}(h \begin{bmatrix} 1 & \\ & 1/p \end{bmatrix}) &= e^{2\pi \text{tr } h \begin{bmatrix} 1 & \\ & 1/p \end{bmatrix}} c_f(h \begin{bmatrix} 1 & \\ & 1/p \end{bmatrix}, b_1) = c_{b_1,f}(Dp^{-1}) = \\ &= (\det \gamma^*)^k c_{b',f}(Dp^{-1}), \end{aligned} \tag{7.2}$$

where we have used the fact that $\epsilon(h) = \epsilon(h \begin{bmatrix} 1 & \\ & p \end{bmatrix}) = \epsilon(h \begin{bmatrix} 1 & \\ & 1/p \end{bmatrix}) = 1$ for h as above (the last equality holding for h such that $h \begin{bmatrix} 1 & \\ & 1/p \end{bmatrix} \in T$). This gives us

$$c_{g_b}(h) = p^2(p+1)(\det \gamma^*)^{-k}(c_{b_1,f}(Dp) + p^{k-2}c_{b_1,f}(Dp^{-1})).$$

The claim now follows from Theorem 7.1. \square

For completeness we also include the analogue of Theorem 7.3 for an inert p . It can be proved in the same way or can be deduced from the results of section 3 of [3].

Let p be an inert prime. The Hecke algebra \mathcal{H}_p is generated by the double cosets $T_p := K_p \text{diag}(p^{-1}, 1, p, 1)K_p$ and $U_p := K_p \text{diag}(p^{-1}, p^{-1}, p, p)K_p$.

Theorem 7.4. *Let p be a rational prime which is inert in F/\mathbf{Q} . There exists a \mathbf{C} -algebra map*

$$\text{Desc}_{\mathcal{B},p} : \mathcal{H}_p \rightarrow \mathbf{T}_p,$$

such that for every $H \in \mathcal{H}_p$ the following diagram

$$\begin{array}{ccc} \mathcal{M}_k & \xrightarrow{H} & \mathcal{M}_k \\ \downarrow \text{Desc}_{\mathcal{B}} & & \downarrow \text{Desc}_{\mathcal{B}} \\ \prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F) & \xrightarrow{\text{Desc}_{\mathcal{B},p}(H)} & \prod_{b \in \mathcal{B}} M_{k-1}(D_F, \chi_F) \end{array}$$

commutes. Moreover, one has

$$(7.3) \quad \begin{aligned} \text{Desc}_{\mathcal{B},p}(T_p) &= p^{-k+4}(p^2 + 1)T_p^2 + p^4 + p^3 + p - 1, \\ \text{Desc}_{\mathcal{B},p}(U_p) &= p^8(T_p^4 + (p+3)p^{k-2}T_p^2 + p^{2k-4}(p^2 + p + 1)). \end{aligned}$$

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